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Tomographic Traveltime Inversion Using Natural Pixels

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SUMMARY

Traditionally in the problem of tomographic traveltime inversion, the model is divided into a number of rectangular cells of constant slowness. Inversion consists of finding these constant values using the measured traveltimes. The inversion process can demand a large computational effort if a high resolution result is desired.

We show in this paper how to use a different kind of parametrization of the model based on beam propagation paths. This parametrization is obtained within the framework of reconstruction in Hilbert spaces by minimizing the error between the true model and the estimated model. The traveltimes are interpreted as the projections of the slowness along the beam paths. Although the actual beam paths are described by complicated spatial functions, we simplify the computations by approximating these functions with functions of constant width and height, i.e., "fat" rays, which collectively form a basis set of natural pixels.

With a simple numerical example we demonstrate that the main advantage of this parametrization, compared with the traditional decomposition of the model in rectangular pixels, is that 2D reconstructed images of similar quality can be obtained with considerably less computational effort. This result suggests that the natural pixels can provide considerable computational advantage for 3D problems.

INTRODUCTION

The process of reconstructing an image using line integrals through it is called tomography. In traveltime tomography the image to be reconstructed is the slowness model $S(\mathbf{r})$. The reconstructed model $\tilde{S}(\mathbf{r})$ is usually represented as a linear combination of functions $\beta_n(\mathbf{r})$ in the form

$$\tilde{S}(\mathbf{r}) = \sum_{n=1}^{M} a_n \beta_n(\mathbf{r}). \qquad (1)$$

The problem consists of determining the unknown coefficients a_n from the measured traveltimes. Once these coefficients have been calculated, the computation of the sum (1) is straightforward.

The kind and number of functions used for expanding the slowness model determine many of the general features of the final image. With the same data set it is possible to obtain different results just because different parametrizations have been used. However, the goal is to obtain a reconstructed model free from these artifacts derived from the parametrization. This means that the selection of the basis function is a critical step in the inversion process and then should be considered more carefully, as described below.

Although the model is usually discretized into rectangular regions of constant slowness (pixels), there is no general criteria for deciding which representation is the best. Some may have clear advantages for solving specific problems, specially if they include prior information about the model. Our selection of the basis function will be based on the minimization of the expression that estimates the norm of the null space of the problem

$$\|S(\mathbf{r}) - \sum_{n=1}^{M} a_n \beta_n(\mathbf{r})\| \qquad (2)$$

where $S(\mathbf{r})$ is the true slowness model. Due to the nature of the measurements in traveltime tomography (integral along beam paths) we show in this paper that the minimum of (2) can be reached when the functions $\beta_n(\mathbf{r})$ describe the beam paths and when M equals the number of measurements available (because there is only one measurement per beam path). In the first part of the paper, this fact is demonstrated within the framework of reconstruction in Hilbert spaces. The remainder of the paper presents some synthetic examples.

RECONSTRUCTION IN HILBERT SPACES

A Hilbert space is a linear space on which an inner product is defined. For example, the inner product for the Hilbert space L^2 of the Lebesque square-integral functions of support Ω is

$$< f(x), \beta(x) > = \int_{\Omega} f(x)\beta(x)dx. \qquad (3)$$

We can assume that the particular function $f(x)$ that we want to estimate belongs to a Hilbert space H. Let's assume also that the information we have about $f(x)$, i.e., data, is a set of inner products of the function $f(x)$ with a finite set of known functions $\beta_m(x) \in H$

$$d_m = < f(x), \beta_m(x) > \qquad m = 1, ..., N. \qquad (4)$$

In this context, the data can be interpreted as the projections of the unknown function $f(x)$ onto the "sampling" functions $\beta_m(x)$.

If F_1 is a closed linear subspace of the Hilbert space H, then $H = F_1 \oplus F_1^{\perp}$ (Berberian, 1976), where $F_1^{\perp}$ is called the orthogonal complement of F_1. From the projection theorem (Stakgold, 1979), we can always decompose $f(x)$ into $f_1(x) + f_2(x)$ where $f_1(x) \in F_1$ and $f_2(x) \in F_1^{\perp}$. $f_1(x)$ is called the orthogonal projection of $f(x)$ in F_1. If we assume that the functions $\beta_n(x)$ form a basis of the space F_1, we can write

$$f(x) = \sum_{n=1}^{N} a_n \beta_n(x) + f_2(x). \qquad (5)$$

We can understand the meaning of the function $f_2(x)$ by multiplying both sides of (5) by $\beta_m(x)$ and integrating in Ω

$$d_m = \sum_{n=1}^{N} a_n < \beta_n(x), \beta_m(x) > + < f_2(x), \beta_m(x) > . \qquad (6)$$

Since $< f_2(x), \beta_m(x) >= 0$, we can say that $f_2(x)$ contains the information about $f(x)$ that does not affect the measurements made by the sampling functions $\beta_m(x)$. Finally, the estimate $\tilde{f}(x)$ of $f(x)$ can then be written as

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$$\tilde{f}(x) = f_1(x) = \sum_{n=1}^{N} a_n \beta_n(x), \qquad (7)$$

where the coefficients a_n are calculated from the forward equation for the data d_m

$$d_m = \sum_{n=1}^{N} a_n < \beta_n(x), \beta_m(x) > \qquad m = 1, ..., N. \qquad (8)$$

This same result for $\tilde{f}(x)$ can be obtained through minimization of the norm $||f_2(x)||$ with respect to the unknown coefficients a_n (Darling et al., 1983),

$$min\ ||f_2(x)||^2 = min\ ||f(x) - \sum_{n=1}^{N} a_n \beta_n(x)||^2. \qquad (9)$$

For this reason, the estimate $\tilde{f}(x)$ is called the minimum norm estimate of the unknown function $f(x)$. This estimate $\tilde{f}(x)$ is unique and consistent with the data (Eqn. (8)). It is also strongly related to the way the modeled data are generated, because the unknown function $f(x)$ is expressed as a linear combination of the sampling functions $\beta_m(x)$ used to compute the forward modeling (Eqn. 4). This means that each experiment will suggest "naturally" the reconstruction procedure which produces the minimum norm solution when the problem is linear.

TOMOGRAPHIC TRAVELTIME INVERSION

The traveltime along a ray l_m in a medium where the slowness is $S(x, y)$, is traditionally given as

$$t_m = \int_{l_m} S(x, y) dl_m \qquad m = 1, ..., N, \qquad (10)$$

where dl_m is the incremental distance along the ray path l_m. In general, the ray path depends on the slowness distribution. For sake of simplicity, let's assume that the variations in the slowness are just a few precent. Then we can safely consider that the ray paths are straight lines and independent of the slowness. The general case will be discussed later.

Although the expression (10) simplifies the mathematics considerably, it fails to convey the fact that the traveltimes between two points are affected by velocities in the region called the Fresnel zone, which is infinitely narrow only when the wavelength λ is infinitely small, $\lambda \rightarrow 0$ (Nolet, 1987). To account for the finiteness of this effect, we can say that the traveltime between two points can be better described by the equation

$$t_m = \int_{\Omega} S(x, y) \phi_m(x, y) dx dy, \qquad (11)$$

where $\phi_m(x, y)$ is a two dimensional function or "beam" of finite support centered along the ray path and Ω is the support of $S(x, y)$. The functions $\phi_m(x, y)$ can be interpreted as the wavepaths introduced by Woodward (1989).

With the forward modeling equation written in this way, the estimation of the slowness from the traveltimes can be seen as a reconstruction problem in a Hilbert space where the inner product is defined by (11). According to (7), the minimum norm estimate of the slowness $S(x, y)$ is

$$\tilde{S}(x, y) = \sum_{n=1}^{N} a_n \phi_n(x, y), \qquad (12)$$

where N is the number of traveltimes.

From Eqn. (8), we find that the coefficients a_n can be calculated through the system of equations

$$t_m = \sum_{n=1}^{N} a_n < \phi_n(x, y), \phi_m(x, y) > \qquad m = 1, ..., N \quad (13)$$

where

$$< \phi_n(x, y), \phi_m(x, y) > = \int_{\Omega} \phi_n(x, y) \phi_m(x, y)\ dx dy. \qquad (14)$$

In contrast with the traditional reconstruction using square pixels as basis function, the reconstruction described above is based on a discretization of the model along the beam paths. The discretization along the beam paths comes from the fact that they are the regions sampled with each measurement in

Natural Pixels

As a first approximation, we can describe the basis function $\phi_i(x, y)$ as functions of width λ' and heigth $1/\lambda'$

$$\phi_i(x, y) = \begin{cases} 1/\lambda' & \text{if } (x, y) \text{ is in the region of width} \\ & \lambda' \text{ centered along the ray path } i \\ 0 & \text{otherwise.} \end{cases} \qquad (15)$$

Therefore, the matrix coefficients $M_{nm} = < \phi_n(x, y), \phi_m(x, y) >$ are

$$M_{nm}\lambda'^2 = \begin{cases} \textit{area of the beam path} & \text{if } m = n \\ \textit{area of the intersection} & \text{if } m \neq n. \end{cases} \qquad (16)$$

A natural pixel for a single ray is shown in Fig. 1. Even when the rays curve or when reflections are included, the natural pixels are "tubes" centered on the ray path.

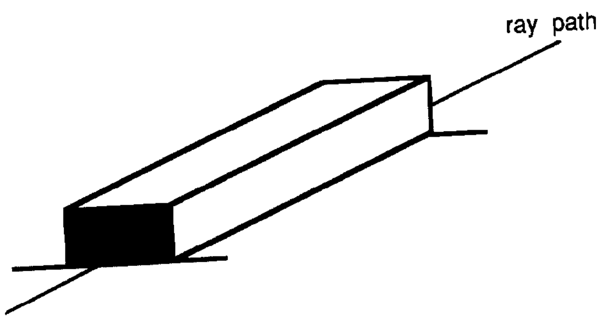

Figure 1: Natural pixel for a single ray.

Buonocore, et al., (1981) and Buonocore (1981), without working within the framework of reconstruction in Hilbert spaces, define an estimator identical to (12) and call it "natural pixel" decomposition of the two dimensional image, where the natural pixels are the functions $\phi_m(x, y)$. They study extensively the properties of such a reconstruction and the theoretical advantages of it compared with the traditional reconstruction using square pixels.

An example of a set of natural pixels is shown in Fig. 2, for the case of a cross borehole geometry in a medium of constant slowness.

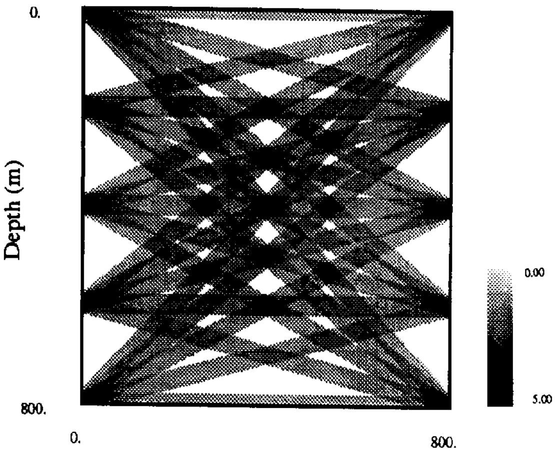

Figure 2: Natural pixels in a constant slowness medium for a cross-well configuration of five sources and five receivers.

To this point, the inversion is strictly linear, this is, the sampling functions do no depend on the slowness. This is analogous to Fourier reconstruction where the sampling functions (complex exponentials) do not depend on the properties of the unknown. No iterations are needed after the estimate is found. This is not the situation in traveltime tomography where the sampling functions may strongly depend on the unknown slowness. The problem is usually solved as a sequence of linearized steps. Natural pixels can be used in each step. Since the ray paths change from one iteration to another, the parametrization along the natural pixels adapts progressively to the estimated model.

NUMERICAL EXAMPLES

We will now show synthetic inversion examples comparing natural pixels and square pixels as basis functions. Our aim is to compare the results of the inversion when both are used with the same data set. This goal can be achieved with synthetic data for a cross borehole geometry generated from the model shown in Fig. 3. The example is simplified considerably by assuming that the slowness contrast between the circular disc ($S = 2.02$) and the background ($S = 2.00$) is 1%. Therefore, straight rays adequately describe the propagation of the energy in the medium.

The data are generated from strip integrals across the model of Fig. 3. For these examples 289 traveltimes were computed, which corresponds to the 17 sources and 17 receivers used. Another simplification is made assuming that the width of the strips $\lambda' = 40$ m is the same during both the forward modeling and the inversion.

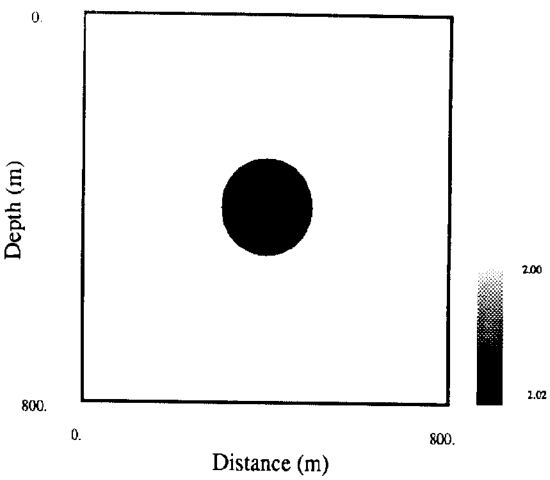

Figure 3: Slowness perturbation. 17 sources are located on the right hand side of the model and 17 receivers are located on the opposite side. The radius of the disc is $r = 100$ m. The width of the natural pixels is $\lambda' = 40$ m. The vertical separation between adjacent sources and/or receivers is 50 m.

When the model is discretized into square pixels the estimate of the slowness is obtained after solving a system of linear equations where the matrix coefficients represent the area of the intersection of the strip with each pixel. We are going to solve this system and the one obtained with the natural pixels (Eqn. 13) using the LSQR variant of the conjugate gradient method (Nolet, 1987) that has been proved to be faster than SIRT methods (Nolet, 1985; Van der Sluis and Van der Vorst, 1987).

Fig. 4 shows the results of the inversion when the model is discretized into square pixels. The starting model has a constant slowness $S_0(x, y) = 2$. The inversion produces directly the slowness value in each pixel, and therefore, reducing the size of the pixels (for better resolution) increases the number of model parameters and consequently the size of the system of equations to solve. In this example, the size of the system of equations solved is 289 X 25921 (grid size = 161 X 161).

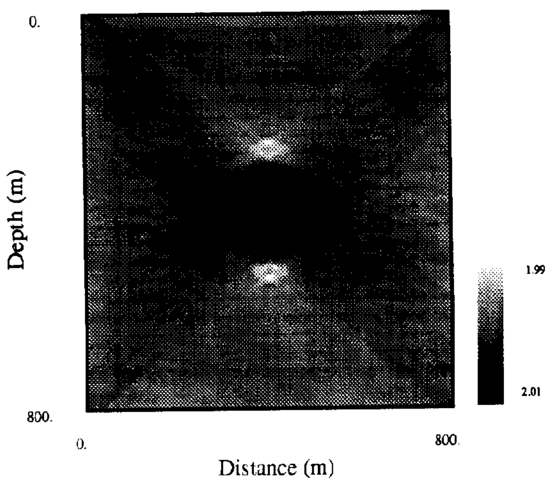

Figure 4: Inversion when a grid of 161 X 161 square pixels is used.

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The result of the inversion using natural pixels is shown in Fig. 5. This image is represented with a grid identical to the one used in Fig. 4 (161 X 161) and then, both results can be compared directly. The system of equations solved with the natural pixels is 289 X 289 and these dimensions are independent of the level of resolution of the image.

Both images look almost identical in terms of resolution. There are no artifacts due to a coarse sampling of the model. The main difference between the two solutions is related with the smoothness of the image. The reconstruction with the square pixels produces an slightly smoother image than the reconstruction with the natural pixels.

However, remember that although the quality of the inversion is basically the same for both basis functions, the computational effort necessary in the whole process is roughly two orders of magnitude smaller when natural pixels are used and both images are densely sampled with the same number of points.

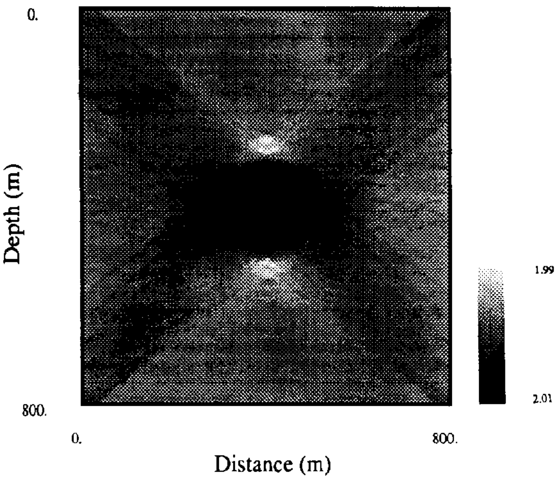

Figure 5: Inversion when the model is discretized in natural pixels. The image is displayed in a grid of 161 X 161 cells.

CONCLUSIONS

We have shown that the natural pixels provide an efficient way of discretizing the slowness model in the problem of traveltime tomographic inversion. In the examples studied, images of similar quality were obtained using natural pixels compared with the traditional reconstruction of square pixels. The main advantage of the natural pixels is that the number of model parameters needed is two orders of magnitude smaller, which means a proportional reduction on the computational effort.

The number of natural pixels equals the number of data points. It means that the number of model parameters in the inversion remains *constant* for a fixed amount of data, regardless the spatial dimensions of the problem or the resolution of the display. Consequently, the natural pixels provide a direct procedure for inversion in three dimensions, problems that can be computationally impossible to attack if the model is described with orthogonal three dimensional pixels (boxes).

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